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RANDOM TIME CHANGE AND RECURRENT BOUNDARY

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The purpose of this paper is to study some properties of random time changes in recurrent potential theory. In particular we show that the Martin recurrent boundary is not invariant under a random time change. We then obtain a characterization of random time change destroying a boundary point. We also give some complement about the recurrent boundary connected with "special additive functionals". We have for example a representation at the boundary of solutions of the Poisson's equation $\eta(I-U_1)=-U_1(x,\cdot)$ by using local time at x.

Recurrent Markov processes	Harris condition
Martin boundary	random time change
Poisson's equation	additives fonctionnals
potential theory	

1. Introduction

The purpose of this paper is to study some properties of random time changes in recurrent potential theory. In particular we shall see that the Martin recurrent boundary introduce in [7] is not invariant under a random time change. We shall obtain a characterization of random time change destroying a boundary point. We We shall also give some complement about the recurrent boundary connected with "special additive functionals". We have for example a representation at the boundary of solutions of the Poisson's equation $\eta(I-U_1)=-U_1(x,\cdot)$ by using local time at x.

The first part conduct to the concept of "A-strongly special" functions which are a localization of special function of Neveu. This notion connected with the boundary by "s-strongly special function" (s belonging to the boundary) will give in the second part an example of random time change destroying locally the boundary.

The second part is devoted to the Martin boundary. This work is the continuation of [7]. So we use the same notations. In all the paper $(U_\alpha)_{\alpha>0}$ will be a Markov resolvent on the state space $(E,\mathscr{E})$. All functions are supposed to be measurable. For each positive and bounded function h, we use the Neveu tabou kernel,

$$U_h=\sum_{n\geq 0}(U_\alpha M_{\alpha-h})^n U_\alpha$$

when $\alpha > \sup_E h$ and M_k is the kernel of multiplication by the function k. We write $h \in \mathscr{E}_+$ (resp. $h \in b\mathscr{E}_+$) to say that h is a positive function (resp. bounded and positive). They satisfy the resolvent equation:

$$k \leq h, \qquad \sum_{n \geq 0} (U_h M_{k-h})^n U_h = U_k.$$

We shall assume in the whole paper that $(U_\alpha)_{\alpha > 0}$ is a *Harris resolvent*; i.e. there exist a positive σ-finish measure m such that

$$(\forall A \in \mathscr{E}) \quad (m(A) > 0 \Rightarrow U_A(A) = 1).$$

It is well known that Harris condition implies the existence of a unique invariant measure μ, up to a multiplicative constant, and that Harris condition holds for μ. When it will exist we shall denote by $X = (X_t, \mathscr{F}_t, \theta_t, P_x)$ the Markov process of resolvent $(U_\alpha)_{\alpha>0}$. When we shall need topological hypothesis E will be a locally compact set and the borel sets. Then we shall always suppose that μ charges all open sets.

2. Random time change and A-special functions

2.1. Random time change

We introduce random time change and first generalize the "tabou" kernel of Neveu to all functions of $\mathscr{E}_+$.

Proposition 2.1. *For all functions $h \in \mathscr{E}_+$, $f \in \mathscr{E}_+$ and for all sequences (h_n) of functions of $b\mathscr{E}_+$ such that $h_n \uparrow h$, the decreasing limit of $U_{h_n}(f)$ is independent of (h_n). This defines a unique kernel U_h. If $(U_\alpha)_{\alpha>0}$ is associated to a Markov process X,*

$$U_h(f)(x) = E_x \int_0^{T_h} \exp\left(-\int_0^t h(x_s)\, \mathrm{d}s\right) f(x_t)\, \mathrm{d}t,$$

where

$$T_h = \inf\left\{t > 0 \,\middle|\, \int_0^t h(x_s)\, \mathrm{d}s = +\infty\right\}$$

and for all functions $k \leq h$ of

$$\sum_{n \geq 0} (U_h M_{k-h})^n U_h = E_x \int_0^{T_h} \exp\left(-\int_0^t k(x_s)\, \mathrm{d}s\right) f(x_t)\, \mathrm{d}t.$$

Proof. Take $k \in b\mathscr{E}_+$ such that $k < \lim_n h_n$, then $U_k(f) = \lim_n U_{h_n}(f)$ for all $f \in b\mathscr{E}_+$ and so the limit does not depend on (h_n). If X is the process associate to $(U_\alpha)_{\alpha>0}$ for all bounded and positive functions $k < h$, we can write

$$(U_h M_{h-k})^n U_h(f)(x) = E_x \int_0^{T_h} f(x_t)\,\mathrm{d}t' \exp\left(-\int_0^t h(x_s)\,\mathrm{d}s\right)$$

$$\times \int_0^t (h-k)(x_{t_1})\,\mathrm{d}t_1 \int_{t_1}^t \cdots \int_{t_{n-1}}^t (h-k)(x_{t_n})\,\mathrm{d}t_n.$$

It is easy then to obtain the formula of the proposition.

It results from the proposition that in general if $k \leqslant h$ we have only $U_k \geqslant \sum_{n\geqslant 0} (U_h M_{k-h})^n U_k$. But for λ and μ in $\mathbb{R}_+$ we have always $U_{\mu h} = \sum_{n\geqslant 0} (\lambda-\mu)^n U_{\lambda h}^{n+1}$ ($\lambda \geqslant \mu$). So by setting

$$^hU_\lambda = U_{\lambda h} M_h$$

we define a sub Markovian resolvent $(^hU_\alpha)_{\alpha>0}$.

Definition 2.2. The passage τ_h from resolvent $(U_\lambda)_{\lambda>0}$ to resolvent $(^hU_\lambda)_{\lambda>0}$ is called the *random time change associate* to the function h. We shall write $\tau_h[(U_\lambda)_{\lambda>0}] = (^hU_\lambda)_{\lambda>0}$.

If $(U_\alpha)_{\alpha>0}$ is the resolvent of the Markov process X, then $(^hU_\lambda)_{\lambda>0}$ is the resolvent of the process $^hX = (X_{\tau_h(t)}, \mathscr{F}_{\tau_h(t)}, \theta_{\tau_h(t)}, P_x)$, and $\tau_h(t)$ is the stopping time defined by

$$t = \int_0^{t_h(t)} h(x_s)\,\mathrm{d}s.$$

The terminal time T_h is the explosion time of the process ^hX.

It comes from what we just have said that the transform by a random time change of a Harris resolvent is not necessarily a Harris resolvent. We shall consider a set of functions h such that $(^hU_\alpha)_{\alpha>0}$ is still Harris. Let us recall for that, that, we denote by $\mathscr{S}$ ($\mathscr{S}_\mathrm{b}$, $\mathscr{S}_1$) the set of Neveu's special functions with respect to $(U_\alpha)_{\alpha>0}$ (see [2]) (special bounded, and bounded by 1):

$$\mathscr{S} = \left\{ f \in \mathscr{E}_+ \mid \forall h \in b\mathscr{E}_+ \sup_E U_h(f) < \infty \right\}.$$

When $(^hU_\alpha)_{\alpha>0}$, for $h \in \mathscr{E}_+$, will be Harris we shall write S^h for the set of special function with respect to $(^hU_\alpha)_{\alpha>0}$ (and also $\mathscr{S}^h_\mathrm{b}$, $\mathscr{S}^h_1$).

We denote N (resp. N^h) the set of change with respect to $(U_\alpha)_{\alpha>0}$ (resp. $(^hU_\alpha)_{\alpha>0}$):

$$N = \{ f \in L^1(\mu) \mid \mu(f) = 0, |f| \in \mathscr{S} \}.$$

Recall at last that a kernel W is a recurrent potential kernel of (U_α) if

$$\forall f \in N, \quad W|f| < \infty, \tag{1}$$

$$\forall f \in N, \forall \alpha \in \mathbb{R}_+, \quad U_\alpha f + \alpha U_\alpha W f = Wf. \tag{2}$$

It is well known that if W is such a kernel all recurrent potential kernel are given by $W' = W + \Phi \otimes \mu + 1 \otimes \nu$ (Φ a finite function, ν a measure finite on $\mathcal{S}$).

Then we have

Proposition 2.3. *Let $h \in \mathscr{E}_+$ such that $\mu(h) > 0$ and suppose there exist*

$$k \in b\mathscr{E}_+,\ k \leq h \quad \text{with } U_k = \textstyle\sum_{n \geq 0} (U_h M_{h-k})^n U_h. \tag{2.3.1}$$

Then the following properties hold:

(1) $U_h(h) = 1$, $h \cdot \mu U_h = \mu$, *for all* $h' \leq h$, *if* h' *satisfies* (2.3.1), *then* $U_{h'} = \sum_{n \geq 0} (U_h M_{h-k})^n U_h$.

(2) *The resolvent* $({}^hU_\alpha)_{\alpha>0}$ *is Harris, its invariant measure is* $h \cdot \mu$.

(3) $(f \in \mathcal{S}^h) \Leftrightarrow (fh \in \mathcal{S})$.

(4) *If* W *is a recurrent potential kernel for* (U_α), WM_h *is also one for* $({}^hU_\alpha)_{\alpha>0}$.

(5) *If* $(U_\alpha)_{\alpha>0}$ *is the resolvent of a process* X, *then the condition* (2.3.1) *is equivalent for* $h \in \mathscr{E}_+$ *to*

$$P_x[T_h = +\infty] = 1 \quad \text{for all } x \in E. \tag{2.3.2}$$

If $\mathcal{T}$ *is the set of functions satisfying* (2.3.1), $\mathcal{T}$ *is a convex cone, stable on the left, and stable by multiplication by functions of* $b\mathscr{E}_+$. *The function of* $\mathcal{T}$ *are* μ *finite, almost everywhere.*

We write ${}^hU_k$ for the "tabou" kernel of k related to $({}^hU_\alpha)_{\alpha>0}$. Then the proof of the proposition lays on the lemma:

Lemma. *For all* h *in* $\mathcal{T}$ *and* k *in* $\mathscr{E}_+$, ${}^hU_k = U_{hk} M_h$.

The lemma results from the fact that if k is a bounded positive function and α a real, $\alpha \geq h$, then

$${}^hU_k = \sum_{n \geq 0} [({}^hU_\alpha M_{\alpha-k})^n]\,{}^hU_\alpha = \sum_{n \geq 0} (U_{h\alpha} M_{\alpha h - hk})^n U_{h\alpha} M_h = U_{hk} M_h.$$

Now for $k \in \mathscr{E}_+$ the result comes from ${}^hU_k \lim \downarrow {}^hU_{k_n}$, where (k_n) is an increasing sequence of bounded positive functions with $\lim_n \uparrow k_n = k$.

Proof of Proposition 2.3. The equivalence between (2.3.1 and (2.3.2) results from Proposition 2.1. We now prove that $({}^hU_\alpha)_{\alpha>0}$ is Harris under condition (2.3.1). Since $(U_\alpha)_{\alpha>0}$ is Harris, there exist a strictly positive function $k_0 < 1$ such that $\mu(k_0) > 0$ $U_{k_0} \geq 1 \otimes \mu$. The function

$$k_1 = \frac{k_0}{h} 1_{[h>1]} + k_0 1_{[h<1]}$$

is strictly positive on E and satisfies $k_1 \leq k_0 \leq 1$, $hk_1 \leq k_0$. Then $\mu(hk_1) > 0$ and $U_{hk_1} \geq U_{k_0} \geq 1 \otimes \mu$. This proves that

$$U^h_{k_1} \geq 1 \otimes h\mu.$$

And so, from [2], the resolvent $({}^hU_\alpha)_{\alpha>0}$ is Harris, its invariant measure being $h\cdot\mu$ from

$$\mu = k\cdot\mu U_k = k\cdot\mu[U_h + U_kM_{h-k}U_h] = h\cdot\mu U_h \quad (k\in b\mathscr{E}_+,\ k\leq h)$$

as h in $\mathscr{T}$.

We shall prove now that $f\in\mathscr{S}^h$ if and only if $f\in\mathscr{S}$. Suppose that $hf\in\mathscr{S}$, then for all function k such that $\mu(hk)>0$, ${}^hU_k(f)=U_{hk}(hf)$ is bounded and so $f\in\mathscr{S}^h$. Conversely, there exists a positive real α such that $\mu\,[0<h<\alpha]>0$. Take k_0 a strictly positive function with $U_{k_0}\geq 1\otimes\mu$. The function $k_1=(1/\alpha)k_0 1_{[0<h<\alpha]}$ satisfies $\mu(hk_1)>0$ and $hk_1\leq k_0$, so

$$U_{k_0}(hf)\leq U_{hk_1}(hf) = {}^hU_{k_1}(f)$$

and the right side is a bounded function since $f\in\mathscr{S}^h$. So $hf\in\mathscr{S}$. The other properties are easily computed.

Remark. Take k a positive function such that $\mu(k)>0$, $U_k\geq 1\otimes\mu$. We can construct the positive recurrent potential kernel associated

$$W=\sum_{n\geq 0}[(U_k-1\otimes\mu)\cdot M_k]^n(U_k-1\otimes\mu).$$

Then for all $h\in\mathscr{T}$ we have

$$U_h + U_hM_hW = W + \frac{1}{\mu(k)}U_h(k)\otimes\mu,$$

$$U_h + WM_hU_h = W + \frac{1}{\mu(k)}1\otimes k\cdot\mu U_h,$$

which are the relations given by Neveu for functions of $b\mathscr{E}_+$.

We also have the following corollary:

Corollary. *If h and h' belong to $\mathscr{S}$ and if $|h-h'|\in\mathscr{S}$, then $\mathscr{S}_b^h=\mathscr{S}_b^{h'}$.*

We also may compose random time change as follows:

Lemma. *Let $h, h'\in\mathscr{T}$ and $k\in\mathscr{E}_+$. If $h'=kh$, then $k\in\mathscr{S}^h$ and $\tau_k[({}^hU_\alpha)]=({}^{h'}U_\alpha)$. We shall write $\tau_{h'}=\tau_k\circ\tau_h$.*

As we can see, the notion of random time change can be done easily with the algebraic technics of Neveu operators U_h. Of course more general random time changes are constructed with additives functionals as it is written in [1], the particular case of functions is closed to the notion that we are going to introduce in Section 2.2. Just before we shall make a last remark.

We have considered, for function h of $\mathcal{T}$, the set $\mathcal{S}^h$ of special functions for $({}^hU_\alpha)_{\alpha>0}$. If $A=\{h>0\}$ it is normal to only consider for $({}^hU_\alpha)_{\alpha>0}$ the state space $E\backslash A^c=A$. For all x the measure ${}^hU_\alpha(x,\cdot)$ does not change A^c. But then, the natural definition of special function should be the functions $f:E\to\mathbb{R}_+$ such that for all $k\in b\mathscr{E}_+$ $\sup_A {}^hU_k(f)<\infty$. In fact the following proposition shows that there is nothing new from $\mathcal{S}^h$:

Proposition 2.4. *Let $h\in\mathcal{T}$, for all functions $f\in\mathscr{E}_+$, be bounded on $A=\{h>0\}$. Then the following statements are equivalent:*

(1) $(\forall k\in\mathscr{E}_+,\ \mu(hk)>0)(\sup_A {}^hU_k(f))<\infty)$.

(2) $f\in\mathcal{S}^h$.

Proof. There is just to prove (1)$\Rightarrow$(2). So take $k\in\mathscr{E}_+$, $k\leq 1$ and $\mu(hk)>0$. Then $hk\leq h$ and

$$\begin{aligned} U_{hk}(hf) &= U_h(hf)+U_h(M_{h-hk}U_{uk}(hf))\\ &\leq \Big(\sup_A f\Big)U_h(h)+\Big(\sup_A U_{hk}(hf)\Big)U_h(h-hk).\end{aligned}$$

This gives $\sup_E {}^hU_k(f)<\infty$ and so $f\in\mathcal{S}^h$.

Remark. Suppose that (U_α) is the resolvent of a Markov process X. From Markov property we easily prove Proposition 2.4:

$$\begin{aligned} U_{hk}(hf)(x) &= E_x\int_T^{+\infty}\exp\Big(-\int_0^t (hk)(x_s)\,\mathrm{d}s\Big)h(x_t)f(x_t)\,\mathrm{d}t\\ &= E_x\Big[E_{x_T}\Big(\int_0^t \exp\Big(\int_0^t (hk)(x_s)\,\mathrm{d}s\Big)h(x_t)f(x_t)\,\mathrm{d}t\Big]\\ &= E_x[U_{hk}hf(x_T)]<\sup_A U_{hk}(f),\end{aligned}$$

Where $T=\inf\{t>0,\ x_t\in A\}$.

2.2. A-strongly special functions

We are now going to introduce a notion which has been first studied by Numelin [3] under the name of "f-strongly uniform" set but which is in fact a localization of Neveu's special functions. By the way of random time change it is easy to obtain many properties of those functions.

Definition 2.5. Let $f\in\mathscr{E}_+$ and $A\in\mathscr{E}$ with $\mu(A)>0$. The function f is said to be *A-strongly special* if for each function $h\in\mathscr{E}_+$, $\mu(h)>0$ $\sup_A U_h(f)<\infty$.

Clearly all special functions are "A-strongly special" for all $A\in\mathscr{E}$. Now one of the most important property of special bounded function h is [2] the fact that $U_{h'}\geq 1\otimes\mu'$ when μ' is a measure equivalent to μ. This turns as follows:

Proposition 2.6. *Let f be a strongly special function. If f is bounded and if $\mu(f)>0$, there exist a measure $\mu' \sim \mu$ such that*

$$U_f \geq 1_A \otimes \mu'.$$

The proof is the same as the proof of the proposition of Neveu [2]. If we take $\theta \in]0, 1/\|f\|_\infty[$, we obtain in the same way

$$U_{\theta f}[\theta(1-f)a] \geq \varepsilon(1-\theta)^{U_B(f)}$$

and so on A,

$$(1-\theta)^{U_B(f)} \geq (1-\theta)^{\sup_A U_B(f)} = C,$$

so

$$U_{\theta f} \geq c\varepsilon \otimes b\mu \quad \text{on } A.$$

The fact that we can remove θ means that we may do the same for U_α:

$$[U_\alpha(\alpha>1) \text{ instead of } U_1.]$$

Proposition 2.7. *For a function $f \in \mathscr{E}_+$ and a part $A \in \mathscr{E}$ such that $\mu(A)>0$, the following statements are equivalent*:

(1) *f is 'A-strongly special'.*

(2) *The function $U_A(f)$ is bounded on A and belongs to $\mathscr{S}^A$.*

(3) *For one function $h_0 \leq 1_A$, strictly positive on A with $U_{h_0} \geq 1 \otimes \mu$, $\sup_A U_{h_0}(f) < \infty$.*

(4) *Let $h \in \mathscr{E}_+$ such that $\mu(h)>0$, $U_h \geq 1 \otimes \mu$ and put $W = \sum_{n \geq 0}[(U_h - 1\otimes\mu)M_n]^n(U_h - 1 \otimes \mu)$. Then $f \in L^1_+(\mu)$ and $\sup_A W(f) < \infty$.*

Proof. (1)$\Rightarrow$(2). For all $h \leq 1_A$ with $\mu(h)>0$ we have

$$U_h(f) = U_A(f) + Uh[(1_A - h)U_A f].$$

It follows that $\sup_A U_h[(1_A-h)U_A f] < \infty$. But $U_A(f)$ is bounded on A and as h is equal to zero on A^c we get $hU_A f \leq ch$, where $c = \sup_A U_A f$, so $U_h[hU_A f] \leq cU_h(h) = c$. This proves with Proposition 2.4 that $U_A f \in \mathscr{S}^A$.

(2)$\Rightarrow$(3). Since $U_A(f) \in \mathscr{S}^A$, $\sup_E {}^A U_{h_0}(f) < \infty$, so

$$\sup_E U_{h_0}[(1_A - h_0)U_A(f)] < \infty.$$

But as $\sup_A U_A(f) < \infty$, the resolvent equation gives $\sup_A U_{h_0}(f) < \infty$.

(3)$\Rightarrow$(4). We suppose $h_0 \leq 1_A$, then $U_A(f) \leq U_{h_0}(f)$ and so $U_A(f)$ is bounded on A. Now,

$$\begin{aligned} A_{U_{h_0}}(U_A f) &= U_{h_0}[1_A U_A(f)] \\ &= U_{h_0}(1_A - h_0)U_A(f) + U_{h_0}(h_0 U_A f) \\ &\leq U_{h_0}(f) + \left(\sup_A U_A(f)\right) U_{h_0}(h_0) \end{aligned}$$

It follows that $\sup_A {}^A U_{h_0}(U_A f) < \infty$. This proves that $U_A(f) \in \mathcal{S}^A$. But then $U_A(f)$ is in $L^1_+(1_A \cdot \mu)$ and we may conclude that $f \in L^1_+(\mu)$. Let W be the recurrent potential kernel of Neveu associated to h_0, then we have $Wf \leqslant U_{h_0}(f) + W(h_0 U_{h_0}(f))$ and as $[h_0 > 0] \subset A$,

$$\sup_A Wf \leqslant \sup_A U_{h_0}(f) + \left(\sup_A U_{h_0}(f)\right) Wh_0 < +\infty.$$

Now for another function h, $U_h \geqslant 1 \otimes \mu$, the associate potential kernel W' satisfied $W' \leqslant W + (1/\mu(h_0)) W'(h_0) \otimes \mu$. This gives $\sup_A W'(f) < \infty$.

(4)$\Rightarrow$(1). For every function $h' \in b\mathscr{E}_+$ with $\mu(h') > 0$, we have $U_{h'}(f) \leqslant V(f) + (1/\mu(h)) U_{h'}(h) \mu(f)$ as $f \in L_+(\mu)$. This proves that $\sup_A U_{h'}(f) < \infty$ and so that f is A-strongly special.

Corollary 1. *If A is a special set with respect to (U_α), a function f is A-strongly special if and only if*

$$\sup_A U_A(f) < \infty.$$

Corollary 2. *If f is A-strongly special for $A \in \mathscr{E}$ and bounded on A, then $f \in \mathcal{S}^A$.*

This corollary comes from ${}^A U_k(f) \leqslant U_k(1_A f)$ $(k < 1_A)$ and so $\sup_A {}^A U_k f < \infty$.

Now it is clear that every function of $\mathcal{S}^A$ which is null on A^c is in $\mathcal{S}$. Then we get also:

Corollary 3. *Every function f, A-strongly special, bounded on A and equal to zero on A^c, is special.*

The function f, A-strongly special generated, the space $L^1_+(\mu)$ in the following sense:

Corollary 4. *For all functions $f \in L^1_+(\mu)$, there exists an increasing sequence of special sets (A_n) with $\lim_n \uparrow A_n = E$ (a.e.) such that for each n, f is A_n-strongly special.*

Proof. Take a strictly positive function h satisfying $U_h \geqslant 1 \otimes \mu$. The associated potential kernel verifies $h \cdot \mu W = c \cdot \mu$ $(c \in \mathbb{R}_+)$. So, if $f \in L^1_+(\mu)$, Wf is finite almost everywhere. The sets $A_n = \{Wf \leqslant n\}$ give the result of the corollary.

3. Boundary and random time change

We begin by some complements on the boundary of a Harris resolvent. They use the notion of special additive functionals introduced by Brancovan [1]. We shall use the notations of [7]. In a later section we shall study the influence of a random time change on the boundary.

3.1. Special additive functional and boundary

Let $(U_\alpha)_{\alpha>0}$ be a Harris resolvent, μ its invariant measure. The boundary of $(U_\alpha)_{\alpha>0}$ is defined by the Choquet's representation of h-invariant measures, $h \in \mathcal{S}_1$, see [7]. The h-invariant measures are the σ-finite measures which satisfy,

$$\nu M_{1-h} U_1 = \nu, \quad \nu(h) < \infty.$$

The cone of h-invariant measures is denoted by I_h. Let us remark the following lemma:

Lemma 3.1. *For a σ-finite measure ν and a function $h \in \mathcal{S}_1$, if $\nu(h) < \infty$, the following statements are equivalent*:

(1) $\nu \in I_h$,

(2) $(\forall p > h, p \in \mathbb{R}_+^*)$ $\nu M_{p-h} U_p = \nu$,

(3) $p > 0, p\nu U_{p+h} = \nu$.

Furthermore (3) *is true since it is true for one* $p > 0$.

Proof. (1)$\Rightarrow$(2). If $\nu \in I_h$, ν is a finite measure on special bounded functions and the resolvent equation gives

$$(\forall f \in \mathcal{S}_b) \quad \nu M_{1-h} U_p f = \nu(f) + \nu(1-p) U_p f.$$

(2)$\Rightarrow$(3). Let $p \geqslant h$, from the resolvent equation again

$$(\forall f \in \mathcal{S}_b) \quad \nu(f) = \nu M_{p-h} U_p f = \nu M_{p-h} U_{p+h} f + \nu M_h U_{p+h}(f) = p\nu U_{p+h}(f).$$

As $U_p - U_q = (q-p) U_p U_q$, (3) is true for all p as soon as it is true for one $p > 0$.

(3)$\Rightarrow$(1). Again the resolvent equation gives

$$(\forall f \in \mathcal{S}_b) \quad \nu U_1 f = U_{1+h} f + \nu U_{h+1} U_1 f = \nu(f) + \nu M_h U_1 f.$$

But $U_1 f$ is bounded so $0 \leqslant \nu M_h U_1 f \leqslant \|U_1 f\|_\infty \nu(h) < \infty$ and then

$$(\forall f \in \mathcal{S}_b) \quad \nu M_{1-h} U_1 f = \nu(f).$$

This proves the lemma.

We shall now suppose that $(U_\alpha)_{\alpha>0}$ is the resolvent of a standard Markov process X. In particular $U_\alpha(C_K) \subset C_b$ for all $\alpha > 0$, where C_K and C_b are the sets of continuous functions on E with compact support and continuous and bounded functions. We denote by Φ the set of special additive functionals (S.A.F.) (see [1] for the definition) which associated measure (see [4]) is not equal to zero. For all $A \in \Phi$ we are going to define the set I_A of A-invariant measure.

Definition 3.2. For all S.A.F. A with a none zero measure ν_A, a positive measure ν is said to be *A-invariant* if for all $p \in \mathbb{R}_+^*$

(1) $p\nu U_{p+A} = \nu$,

(2) $p\nu U_{p+A}^A(1) < \infty$

with the notation

$$U_B^A f(x) = E_x \int_0^\infty \exp(-B_t) f(X_t)\, dA_t, \tag{3.2.1}$$

where A and B are two continuous additive functionals (C.A.F.).

We shall introduce a notation in order to simplify the measure of (2) in the definition. For that just remark that if $\nu \in I_A$ and if B is a C.A.F., then for all p, p' of $\mathbb{R}_+^*$

$$p\nu U_{p+A}^B = p'\nu U_{p'+A}^B. \tag{3.2.2}$$

This is just coming from the resolvent equation for additive functionals:

$$p\nu U_{p+A}^B = p\nu U_{p'+A}^B + p\nu U_{p+A}^{p'-p} U_{p'+A}^B \quad (p<p') = p'\nu U_{p+A}^B.$$

Thus we shall write ν^B for the measure of (3.2.2). That is

$$\nu^B = p\nu U_{p+A}^B$$

for one $p>0$.

We just remark also that if A is the S.A.F. defined by

$A_t = \int_0^t h(X_B)\, ds$, the cone I_A is equal to I_h. And if B is the C.A.F.

$B_t = \int_0^t k(X_B)\, ds$, then $\nu^B = k \cdot \nu$.

Now we get the same result as in [7] but generalized to S.A.F.

Proposition 3.3. *Let A and B two S.A.F. whose associated measure are non null. Then there exist a positive constant k which depend only on A and B such that for all A-invariant measures ν*

$$\nu^B(1) \leq k\nu^A(1). \tag{3.3.1}$$

The cones I_A are all isomorphic for $A \in \Phi$. If u_{BA} is the isomorphism from I_A to I_B,

$$\forall \nu \in I_A, [u_{B,A}(\nu)](1) = \nu^A(1).$$

Proof. We have just to prove (3.3.1) when $B \geq A$ (in the other case we shall compose with I_{A+B}). We need to prove first the next lemma:

Lemma. *If A and B are two S.A.F.'s and C is a C.A.F. ($\nu_C > 0$), then for all $\nu \in I_A$,*

$$\nu^{B-A} U_B^C \leq \nu^C. \tag{3.3.2}$$

Proof of the lemma. From the resolvent equation

$$\nu^C = \nu U_{A+1}^C = \nu U_{B+1}^C + \nu U_{A+1}^{B-A} U_{B+1}^C.$$

Now for all C.A.F., C,

$$\nu^C \geq \nu_A + U_{A+1}^{B-A}\left(\sum_{p=1}^{n} (U_{B+1})^p\right)^C \quad (n \in \mathbb{N}), \tag{3.3.3}$$

indeed, by a recurrent argument on $\mathbb{N}$,

$$\nu U_{A+1}^{B-A}\left[\sum_{p=1}^{n} (U_{B+1})^p U_{B+1}^C\right] = \nu U_{A+1}^{B-A} U_{B+C}^C + \nu U_{A+1}^{B-A}\left[\sum_{p=1}^{n} (U_{B+1})^p U_{B+1}^C\right].$$

But setting $C = 1$ the recurrent hypothesis gives

$$\nu U_{A+1}^{B-A}\left[\sum_{p=1}^{n} (U_{B+1})^p\right] < \nu,$$

so

$$\nu U_{A+1}^{B-A}\left[\sum_{p=1}^{n} (U_{B+1})^p U_{B+1}^C\right] \leq \nu U_{A+1}^{B-A} U_{B+1}^C + \nu U_{B+1}^C \leq \nu^C.$$

And the lemma comes when $n \uparrow +\infty$

We may now obtain (3.3.1). Take B a S.A.F. with $B \geq A$. Then from [1] there exist a measure $\mu' \sim \mu$ such that $U_{B'} \geq 1 \otimes \mu'$. It is easy then to prove $U_B^A \geq 1 \otimes \nu'$, where ν' is an equivalent measure to ν_A.

If we take $C = A$ in (3.3.2) we get,

$$\nu^{B-A}(1) \leq \frac{1}{\nu'(1)} \nu^A(1) < \infty,$$

so

$$\nu^B(1) = \nu_{A+1}^B(1) \leq [1 + (\nu'(1))^{-1}]\nu^A(1),$$

and the first part of the proposition is proved.

In order to obtain the isomorphism between the cones I_A we shall show that I_A, (A S.A.F.) is isomorphic to I_B, where B is given by

$$B_t = \int_0^t h(X_s)\,\mathrm{d}s, \quad (h \in \mathcal{S}).$$

First we remark that if $\nu \in I_B$ ($= I_h$), then $\nu(h) \leq k\nu^A(1) < \infty$ so that if $\nu^A(1) = 0$, then ν is null. If we construct ($B \geq A$)

$$u_{B,A} : I_A \to I_B$$

$$\nu \mapsto \nu - \nu^{B-A} U_B,$$

$$u_{A,B} : I_B \to I_A$$

$$\nu \mapsto \nu + \nu^{B-A} U_A,$$

it follows from the results above that $u_{B,A}$ and $u_{A,B}$ are one to one and that $u_{B,A} \circ u_{A,B} = \mathrm{id}_B$, $u_{A,B} \circ u_{B,A} = \mathrm{id}_A$. This ends the proof of the proposition.

Remark. It follows from the proposition that the measures of I_A are Radon's. The representation to the boundary of h-invariant measures can be carried to A-invariant measures. So, if we write $\{U_A(s, \cdot\,); s \in S\}$ the set of extremal measure of $\hat{I}_A = \{\nu \in I_A \mid \nu^A(1) < 1\}$ with the convention that for all $h \in \mathscr{S}$ $u_{h,A}[U_A(s, \cdot\,)] = U_h(s, \cdot\,)$, we get a one to one correspondence $m \to \int_S U_A(s, \cdot\,) m(\mathrm{d}s)$ between the set $\mathscr{M}_+(S)$ of positive and bounded measure on S and I_A. For more details on the boundary S see [7].

Remark. It is enough for $\nu \in I_A$, that relation (1) of Definition 3.1 is satisfied for one $p > 0$.

We are now going to apply the result above to solve the Poisson's equation for measure:

$$\eta(I - U_1) = -\tilde{\nu}_A U_1, \tag{3.4.1}$$

where $\tilde{\nu}_A = (1/\nu_A(1))\nu_A$, A being a S.A.F. Formerly if η is a solution of (3.4.1), $\eta = \eta U_1 - \tilde{\nu}_A U_1$. So we can define for all C.A.F.'s a unique measure $\eta^B = \eta U_1^B - \tilde{\nu}_B U_1^B$. It is easy to see that η^B is a Radon's measure and moreover $\eta^B(f) < \infty$ for all $f \in \mathscr{S}_b$. Then we get the proposition

Proposition 3.4. *Let A be a S.A.F. with $\nu_A > 0$ and $U_A \geq 1 \otimes \mu$. All the solutions η of the Poisson equation*

$$\eta(I - U_1) = -\tilde{\nu}_A \cdot U_1 \quad (\tilde{\nu}_A = (1/\nu_A(1))\nu_A) \tag{3.4.1}$$

which are such that the measure η^B is finite on $\mathscr{S}_b$, are given by

$$\eta = k[\nu + \tilde{\nu}^A W_A] \quad (k \in \mathbb{R}_+^*)$$

where ν is in I_A and

$$W_A = \sum_{n \geq 0} (U_A^A - 1 \otimes \nu_A)^{n-1}(U_A - 1 \otimes \mu).$$

In particular these solutions η which verify $\eta^A(1) = 1$ are in one to one correspondence with the probability m on S by the formula

$$\eta = \nu_A(1) \int_S [U_A(s, \cdot\,) + U_A^A(s, \cdot\,) W_A] m\,(\mathrm{d}s).$$

(Notation $U_A^A(s, \cdot\,)$ stands for $[U_A(s, \cdot\,)]^A$, $s \in S$)

Proof. It follows first from Proposition 3.3 that $\eta = \nu + \tilde{\nu}^A W_A$ ($\nu \in I_A$) is a solution of (3.4.1), because

$$\eta(\nu + \tilde{\nu}^A W_A)U_1 = \nu + \tilde{\nu}^A U_1 + \tilde{\nu}^A W_A U_1 = \nu + \tilde{\nu}^A W_A + \tilde{\nu}^A(1) \cdot \tilde{\nu}_A U_1.$$

Conversely, take η solution of (3.4.1) with the indicated conditions from $\eta U_1^A = \eta^A + \tilde{\nu}_A U_1^A$, we obtain

$$\eta U_1 = \eta^{A+1} U_{A+1} + \tilde{\nu}^A U_1^A U_{A+1}.$$

This can be written

$$\eta + \tilde{\nu}_A U_1 = \eta^{A+1} U_{A+1} + \tilde{\nu}_A U_1^A U_{A+1}.$$

But $\tilde{\nu}_A U_1 = \tilde{\nu}_A U_{A+1} + \tilde{\nu}_A U_1^A U_{A+1}$, as it is a finite measure on functions of $\mathcal{S}$, we get

$$\eta + \tilde{\nu}_A U_{A+1} = \eta^{A+1} U_{A+1}. \tag{3.4.2}$$

We now multiply the equality by $\sum_{p=0}^{n} (U_{A+1})^p$; from the fact that the measure $\eta(U_{A+1})^p$ is finite on special functions (cf. $\eta(U_{A+1})^p < \eta(U_1)^p$), we get

$$\eta + \tilde{\nu}_A \sum_{p=1}^{n+1} (U_{A+1})^p = \eta^A \sum_{p=1}^{n+1} (U_{A+1})^p + \eta(U_{A+1})^{n+1}.$$

We take $n \uparrow \infty$ and obtain $\eta + (1/\nu_A(1))\mu \geq \eta^A U_A$. We may suppose that for $\eta^A(1) = (1/\nu_1(A))$ we get $\eta > \eta^A[U_A - 1 \otimes \mu]$.

Write $V_A = U_A - 1 \otimes \mu$, for all function f in $\mathcal{S}$, we then have

$$\eta + \eta^A W_A(f) < \eta(f) + \|W_A f\|_\infty \eta^A(1) < \infty.$$

So the measure $\eta + \eta^A W_A$ and $\eta^A V_A + \eta^A V_A W_A$ are both finite on $\mathcal{S}$. And so

$$\eta = \eta - \eta^A V_A + [\eta - \eta^A V_A]^A W_A.$$

The proof will be complete if we verify that $\eta_0 = \eta - \eta^A V_A$ is A invariant measure. From (3.4.2) we obtain, since $\eta = \eta_0 + \eta^A W_A$,

$$\eta_0 + \eta_0^A W_A + \nu U_{A+1} = \eta_0 + \eta_0^A W_A^{A+1} U_{A+1}. \tag{3.4.3}$$

But U_A and W_A are connected by classical relations [1] and so,

$$\eta_0 + \eta_0^A W_A^{A+1} U_{A+1} = \eta_0 U_{A+1} + \eta_0^A W_A + \frac{1}{\nu_A(1)} 1 \otimes \mu U_{A+1}. \tag{3.4.4}$$

All measures in (3.4.3) and (3.4.4) are finite on functions of $\mathcal{S}$ so we may conclude

$$\eta_0 = \eta_0 U_{A+1}$$

which shows that η_0 is A-invariant.

Remark. We have used the fact, easy to prove, that $\eta^{A+B} = \eta^A + \eta^B$. On an other hand we have introduced for $\eta_0 = \eta - \eta^A V_A$ the measure $\eta_0^A = \eta_A - \eta_A V_A^A$, where of cause $V_A^A = U_A^A - 1 \otimes \nu_A$. It is also easy to see that η_0^A is well defined.

There is an interesting application of Proposition 3.4. We make the hypothesis that the measure μ charges the thin open set. Then if in $x \in E$ the local time L_x exists, it is an S.A.F. [1]; its associate measure is ε_x up to a multiplicative constant and it is proved in [1] that we can choose L_x such that $U_{L_x} \geq 1 \otimes \mu$. Then we can apply Proposition 3.4 to obtain a complete generalization of the results of [4] for the discrete case.

Corollary. *Let $x \in E$, where local time L_x exists. Then all solutions of Poisson's equation,*

$$\eta(I - U_1) = -U_1(x, \cdot), \tag{3.4.5}$$

such that for all S.A.F.'s *η^B is a finite measure, are given by*

$$\eta = \nu + \nu^{L_x} W_{L_x},$$

where ν is a L_x-invariant measure. In particular, solutions of (3.4.5) *are in one to one correspondence with positive and finite measure m on S by the formula*

$$\eta = \int_S [U_{L_x}(s, \cdot) + U_{L_x}^{L_x}(s, \cdot) W_{L_x}] m(\mathrm{d}s).$$

Remark. We also can construct the sets I_x of L_x-invariant measure as mentioned in the corollary. These set are then in particular isomorphic when $x \in E$ and we have the complete generalization of the construction of the recurrent boundary for the discrete case of Kemeny shell.

3.2. Extension to the boundary

We now suppose that the boundary S is not empty. We shall consider the notion of h-invariant measure and the set I_h for any function of $\mathscr{S}$ (of course I_h may be trivial). For that we study the extension of U_h to $(E+S) \otimes \mathscr{E}$ for all $h \in \mathscr{T}$.

Proposition 3.5. *Let $h \in \mathscr{T}$. For all $s \in S$ and all special functions $k \leq h$, the formula*

$$U_h(s, \cdot) = U_k(s, \cdot) - U_k(s, \cdot) M_{h-k} U_h \tag{3.5.1}$$

defines a positive measure $U_h(s, \cdot)$ which does not depend of the function k, finite on the functions of $\mathscr{S}_b$ and such that

$$U_h(s, h) \leq 1, \qquad (\forall p > 0) \quad pU_h(s, \cdot) U_{p+h} = U_h(s, \cdot). \tag{3.5.2}$$

Moreover if the measure $U_h(s, \cdot)$ is not equal to zero, then $U_h(s, h) = 1$ and for all special functions $k \leq h$

$$U_k(s, \cdot) = U_h(s, \cdot) + U_h(s, \cdot) M_{h-k} U_k. \tag{3.5.3}$$

It follows from the proposition that for all $h \in \mathcal{T}$,

$$U_h(s, h) \equiv 0 \text{ or } 1.$$

We begin by a lemma.

Lemma. *Let $h \in \mathcal{C}$, $k \in \mathcal{S}_b$, $k \leq h$. For all k-invariant measures ν,*

(1) $\nu M_{h-k} U_h \leq \nu$, if $\nu' = \nu - \nu M_{h-k} U_h$, *the measure ν' is finite on $\mathcal{S}_b$ and*

(2) $\nu'(h) \leq \nu(k)$,

(3) $p\nu' U_{p+h} = \nu'$ *for all $p \geq \nu$. At last if $k' \in \mathcal{S}_b$, then*

(4) $\nu' = u_{k'h}(\nu) - u_{k'h}(\nu) M_{h-k'} U_h$.

Proof of the lemma. The proof of (1) is the same as the proof of Lemma (I.1) of [7].

Let $k' \in \mathcal{S}_b$ such that $k' \leq k \leq h$, we denote $\nu_1 = u_{k',k}(\nu)$ and the corresponding measure $\nu'' = \nu_1 - \nu_1 M_{h-k} U_h$. Then relation (4) means that $\nu' = \nu''$. To show it we remark that $\nu_1 = \nu'' + \nu_1 M_{h-k'} U_h$ and so

$$\nu + \nu M_{k-k'} U_{k'} = \nu'' + (\nu + \nu M_{k-k'} U_{k'}) M_{h-k'} U_h.$$

Then

$$\begin{aligned} \nu + \nu M_{k-k'} U_{k'} &= \nu'' + \nu M_{h-k} U_h + \nu M_{k-k'} U_h + \nu M_{k-k'} U_{k'} M_{h-k'} U_k \\ &= \nu'' + \nu M_{h-k} + \nu M_{k-k'} U_{k'}. \end{aligned}$$

All those measures are finite on $\mathcal{S}_b$. We may conclude

$$\nu'' = \nu - \nu M_{h-k} U_h = \nu'$$

Now there exists an increasing sequence (k_n) of functions of $\mathcal{S}_b$ with $\lim_n \uparrow k_n = h$. Put $\nu_n = u_{k_n,k}(\nu)$, then $\nu' \leq \nu_n$ and

$$\nu'(h) \leq \lim_n \uparrow \nu'(k_n) \leq \nu_n(k_n) = \nu(k).$$

We now prove (3). If $p > 0$ is given,

$$\nu = p\nu U_{p+k} = p\nu U_{p+k} + \nu U_{h-k} U_{p+k}.$$

Then

$$\begin{aligned} \nu &= p\nu' U_{p+k} + p\nu M_{h+k} U_h U_{p+h} + \nu M_{h-k} U_{p+h} \\ &= p\nu' U_{p+k} + \nu M_{h-k} U_h. \end{aligned}$$

And this gives $\nu' = p\nu' U_{p+h}$.

Proof of Proposition 3.5. From the lemma we get the first part of the proposition. Remark that (3.5.3) follows from (3.5.4), so we have just to prove (3.5.4). For that write $\nu = U_k(s, \cdot)$ and $\nu' = U_h(s, \cdot)$. As ν is excessive for the kernel $M_{h-k} U_h$, its Riesz decomposition is written

$$\nu = \nu' I + M_{h-k} U_k + \lim_n \uparrow \nu (M_{h-k} U_h)^n = \nu_1 + \rho.$$

All those measures are finite on $\mathcal{S}_b$. The resolvent equation shows that ν_1 is k-invariant and so is ρ. But $\nu_1(k) + \rho(k) = \nu(k) = 1$ and as

$$\nu = \nu_1(k) \frac{\nu_1}{\nu_1(k)} + \rho(k) \frac{\rho_1}{\rho(k)}$$

we must have $\nu_1 = 0$ or $\rho = 0$ because ν is extremal. As $\nu_1 \neq 0$ we get $\nu = \nu' I + M_{h-k} U_k$. This is exactly (3.5.4).

We now get the topological properties.

Proposition 3.6. *Let $h \in \mathscr{C}$ be strictly positive and continuous. Then for all $f \in C_K^+(E)$ the application*

$$E + S \to \mathbb{R}_+$$
$$x \mapsto U_h(x, f)$$

is upper semi-continuous. ($C_K^+(E)$ is the set of positive functions on E with compact support.)

Proof. We take $k \in C_K^+(E)$, $k \leq h$, $\mu(k) \neq 0$. Then, for all $x \in E + S$,

$$U_h(x, f) = U_k(x, f) + U_k(x, \cdot)(h - k) U_h(f).$$

But we know from Proposition (II.1) and (II.5) of [7] that $x \to U_k(x, f)$ is continuous on $E + S$, so we have just to prove that $x \to U_k(x, \cdot)[(h-k) U_h(f)]$ is lower semi-continuous on $E + S$.

Let g be a positive bounded and continuous special function on E. Then there exists a sequence (g_n) of $C_K^+(E)$ such that if $g_n \uparrow g$, then $U_k(x, g) = \lim_x \uparrow U_k(x \cdot g_n)$ for all $x \in E + S$. But $U_k(\cdot, g_n)$ is continuous on $E + S$ as we just said and so $U_k(\cdot, g)$ is lower semi-continuous on $E + S$. Now the function $g = (h - k) U_h(f)$ is special, when $f \in C_K^+(E)$ (cf. $g \leq h U_h(f)$), bounded and continuous [7, Proposition (II.1)]. This proves the proposition.

We now get some limit result for U_p ($p \in \mathbb{R}_x^+$).

Proposition 3.7. *For all $p > 0$ and $s \in S$,*

$$\lim_{\substack{x \to s \\ x \in E}} U_p(x, \cdot) = 0$$

for weak-convergence of measure.

Proof. The measure $(U_p(x,\cdot))_{p>0}$ are relatively weakly compact, so we have just to prove that every adherent measure λ of $\{(U_p(x,\cdot)), x\to s\}$ is null. Let $k\in C_K^+(E)$, $k\leq p$, we have $U_k(s,\cdot)=U_k(s,\cdot)M_{p-k}U_p$ and if $x\in E$, $U_k(x,\cdot)=U_p(x,\cdot)+U_k(x,\cdot)M_{p-k}U_p$. If (x_n) is a sequence of E which converges to ρ (in the sense of the topology of $E+S$ defined in [7]), $U_p(x_n,\cdot)$ converges to λ. We get as $U_k(x_n,\cdot)$ weakly converges to $U_k(s,\cdot)$, that for all $f\in C_K^+(E)$,

$$U_k(s,\cdot)[(p-k)U_p(f)]\leq \liminf_n U_k(x_n,\cdot)(p-k)U_p(f).$$

And so $U_k(s,f)+\lambda(f)\leq U_k(s,f)$. This means that for all $f\in C_K^+(E)\lambda(f)=0$ so the measure $\lambda=0$.

Corollary 1. *For all compact sets K of E, $\lim_{x\to s} U_1(x,K)=0$.*

This corollary has the following significance when (U_p) is the resolvent of a standard process. Put for all compact K of E

$$T_K=\inf\{t>0\,|\,X_t\in K\}.$$

Corollary 2. *For all compact sets K of E such that the interior of K is not empty and for all $a\in\mathbb{R}_+^*$,*

$$\lim_{\substack{x\to s\\ x\in E}} P_x[T_K<a]=0.$$

Proof. If the interior of K is not empty, $b=\inf_K U_1(x,K)$ is strictly positive because there exists a continuous function ϕ with $0<\phi<1_K$. But the Harris condition proves that $\mu\ll U_1(x,\cdot)$, so as $\mu(\phi)>0$, $b\geq\inf_K U_1(x,\phi)>0$. Now the Markov property gives

$$\begin{aligned}U_1(x,K)&=E_x\int_{T_K}^{+\infty} e^{-t}1_K(X_t)\,dy\\&=E_x\Big(E_x\Big(e^{-T_K}\int_0^{\infty} e^{-u}1_K(X_{T_K}+u)\,du\Big)\,|\,\mathcal{F}_{T_K}\Big)\Big)\\&=E_x[e^{-T_K}U_1(X_{T_K},K)],\end{aligned}$$

so $U_1(x,K)>bE_x[e^{-T_K}]$ and thus $P_x[T_x<a]\leq e^aU_1(x,K)$. This proves the corollary.

We shall now give an example of a set of functions h for which the set of h-invariant function is not trivial and which contains the set of special functions.

Definition 3.8. A measurable and positive function f on E is called *s-strongly special* for an $s\in S$ if there exist a neighbourhood V of s in $E+S$ such that f is $(V\cap E)$-strongly special.

Proposition 3.9. *Let h be a continuous function on E which is s-strongly special, then the measure* $(U_n(s,\cdot))$ *is non zero.*

Proof. We have that for an equivalent measure μ' to μ, $U_h \geq 1_V \otimes \mu'$ (V is a neighbourhood of s). So $U_h(s,f) = \lim_{x\to s} U_h(x,f) \geq \mu'(f) > 0$ for all $f \in C_K^+(E)$ and this gives the proof.

3.3. Transform of the boundary by random time change

Let h be a special function, then the function 1 is special for the resolvent $({}^hU_\alpha)_{\alpha>0}$ and thus the asociated recurrent boundary S^h is empty. This shows that the boundary is not invariant under a random time change.

For all random time change τ_h we shall denote by S^h the associated boundary and ${}^hI_k$ the cone of k-invariant measures with respect to $({}^hU_\alpha)$, i.e.,

$${}^hI_k = \{\nu \in \mathcal{M}_+(E) | \nu(k) < \infty \; p\nu({}^hU_{p+k}) = \nu \, p > 0\}.$$

Then we have the following result·

Proposition 3.10. *Let h and h' be two functions of* $\mathcal{T}$ *such that* $0 \leq h \leq h'$ *and let* $k \in \mathcal{S}^h$. *Then* $kh/h' \in \mathcal{S}^{h'}$ *(with* $h/h' = 0$ *on* $h = 0$*) and the application*

$$\Phi_{h,h'} : \nu \mapsto \nu M_{1-k} {}^{h'}U_{h/h'}$$

is an injection of ${}^hI_k$ *to* ${}^hI_{kh/h'}$. *All extremal measures of* ${}^h\hat{I}_k$ *have an extremal image in* ${}^{h'}\hat{I}_{kh/h'}$. *(We recall that* ${}^h\hat{I}_k = \{\nu \in {}^hI_k \,|\, \nu(k) \leq 1\}$.*)*

Proof. It is enough to prove the proposition when $h' = 1$. Take $\nu \in {}^hI_k$, and put $\eta = \nu M_{1-k} U_h$, then $\eta M_h = \nu$ and clearly $\Phi_{k,1}$ is an injection from ${}^hI_k$ to I_{kh}. Now

$$\begin{aligned} \eta M_{1-hk} U_1 &= \eta M_{1-h} U_1 + \eta M_{h-k} U_1 \\ &= \nu M_{1-k} U_1 + U_h M_{1-h} U_1 \\ &= \eta, \end{aligned}$$

so $\eta \in I_{hk}$ as $\eta(hk) = \nu(k) < \infty$.

For all measures η of I_{hk} we have $\eta M_{h-hk} U_h \leq \eta$; so if $\nu \in {}^h\hat{I}_k$ is extremal, then if $\eta = \Phi_{h,1}(\nu)$, the relation

$$\eta = \lambda \eta_1 + (1-\lambda)\eta_2, \quad 0 \leq \lambda \leq 1, \, \eta_i \in \hat{I}_{hk}$$

shows that η_i $(i = 1, 2)$ are in $\Phi_{h,1}({}^hI_k)$ and so $\lambda = 0$ or $\lambda = 1$. This proves that η is also extremal.

Remark. We have used the fact that a measure $\eta \in I_{nk}$ belongs to $\Phi_{h,1}({}^hI_k)$ if and only if $\eta = \eta M_{h-hk} U_h$.

Corollary. *If h and h' are two positive functions such that $h \leq h'$ and $h' - h \in \mathscr{S}_b$, then, for all functions $k \in \mathscr{S}^{h'}$, $\Phi_{h,h'}$ is an isomorphism.*

These results must be seen from the point of view of the boundary. If we keep in mind the way we used to construct the recurrent boundary we see that, for example if $h \leq h'$ are two functions of $\mathcal{T}$, if S^h and $S^{h'}$ are the corresponding boundaries of the Harris resolvents $({}^hU_\alpha)$ and $({}^{h'}U_\alpha)$, Proposition 3.10 means that $S^h \subset S^{h'}$. Of course, there is just a condition of compatibility for the notations. That is, if we recall a little the boundary construction:

For all function $h \in \mathcal{T}$: ${}^hI_k$ is the set of k-invariant measures with respect to the Harris resolvent $({}^hU_\alpha)_{\alpha>0}$ and

$$ {}^h\hat{I}_k = \{\nu \in {}^hI_k \mid \nu(k) \leq 1\}. $$

Then, for a fixed $h \in \mathcal{T}$, we write $\{{}^hU_k(s, \cdot\,), s \in S^h\}$, the set of all extremal measures of ${}^h\hat{I}_k$, for every $k \in S^h$. The result of [7] has proved that S^h does not depend on the function k and it is the boundary of $({}^hU_\alpha)_{\alpha>0}$.

Changing now of function h, we know from Proposition 3.10 that if $h, h' \in \mathcal{T}$ are such that $h \leq h'$, $\phi_{hh'}: {}^hI_{hk} \to {}^{h'}I_{hk}$ is an injection, so we shall get that, if $h \leq h'$, then $S^h \subset S^{h'}$ from the following fact: if $\eta \in {}^h\hat{I}_k$ is an extremal measure, so is $\Phi_{hh'}(\eta)$ in ${}^{h'}\hat{I}_{hk/h'}$ and if $\eta = {}^hU_k(s, \cdot\,)$ for one $s \in S^h$, then $\Phi_{hh'}(\eta) = {}^{h'}U_{hk/h'}(s, \cdot\,)$: with the same $s \in S^{h'}$ and $S^h \subset S^{h'}$. This can be summed up in the following proposition.

Proposition 3.11. *Let $(U_\alpha)_{\alpha>0}$ be an Harris resolvent, S its Martin recurrent boundary. Let h and h' be two functions of $\mathcal{T}$, τ_h and $\tau_{h'}$ the associated random time changes and $({}^hU_\alpha) = \tau_h[(U_\alpha)]$, $({}^{h'}U_\alpha) = \tau_{h'}[(U_\alpha)]$ the transformed Harris resolvents, S^h and $S^{h'}$ the corresponding boundaries. Then if $h \leq h'$, $S^h \subset S^{h'}$. In particular: For every acceleration τ_h* (i.e. $h < 1$), *$S^h \subset S$. If h is special $S^h = \emptyset$.*

But when the random time change slackens (i.e. $h > 1$), *then $S^h \supset S$.*

If we gather the results of Propositions 3.5 and 3.10 we already see the following result that we just write for an acceleration:

Proposition 3.12. *Let $0 < h \leq 1$ be a function of $\mathcal{T}$. A point s of S in conserved in the random time change τ_h* (i.e. $s \in S^h$) *if and only if $U_h(s, \cdot\,) = 0$.*

Corollary 1. *Let $h \in \mathcal{T}$, and suppose that $s \in S$ still belongs to S^h. Then*

$$ \lim_{\substack{x \to s \\ X \leftarrow E}} U_h(x, f) = 0, \quad \forall f \in C_K(E). $$

This follows from Proposition 3.6:

$$ \limsup_{\substack{x \to s \\ x \in E}} U_h(x, f) = 0 $$

for all $f \in C_K^+(E)$, so

$$\lim_{\substack{x \to s \\ x \in E}} U_h(x, f) = 0.$$

Corollary 2. *For every s-strongly special function h, $s \notin S^h$.*

5.4. Example of the Brownian motion on $\mathbb{R}$

The resolvent of the Brownian motion has the densities

$$u_p(x, y) = \frac{1}{\sqrt{2p}} \mathrm{e}^{-\sqrt{2p}|x-y|}$$

Then, if h is a positive function, the h-invariant measures are absolutely continuous and the densities a solution of $f'' = hf$. The converse is easy to obtain if we do not forget the condition $\nu(h) < \infty$. For example if $h = 1_{[0,1]}$, there are two extremal measures in $\hat{I}_h$ of densities

$$f_1(x) = \begin{cases} (-\sqrt{2}\,\mathrm{Sh}\,\sqrt{2})x + \mathrm{Sh}\,\sqrt{2}, & x < 0, \\ \mathrm{Ch}\,\sqrt{2}\,(x-1), & 0 < x < 1, \\ 1, & x > 1, \end{cases}$$

$$f_2(x) = \begin{cases} 1, & x < 0, \\ \mathrm{Ch}\,\sqrt{2}\,x, & 0 < x < 1, \\ (\sqrt{2}\,\mathrm{Sh}\,\sqrt{2})x + \frac{3}{2}\mathrm{e}^{-\sqrt{2}} - \frac{1}{2}\mathrm{e}^{\sqrt{2}}, & x > 1. \end{cases}$$

This gives the boundary points $+\infty$ and $-\infty$.

If $h = 1_{[0,\infty[}$, there is only one extremal measure of density:

$$f(x) = \begin{cases} -x + \frac{1}{2}, & x < 0, \\ \dfrac{\mathrm{e}^{-\sqrt{2}x}}{\sqrt{2}}, & x > 0. \end{cases}$$

So $U_h(+\infty, \cdot) = 0$ and $U_h(-\infty, \cdot) > 0$, so $S^h = \{+\infty\}$.

We may also get $(+\infty)$-strongly special functions and $(-\infty)$-strongly special functions. For that we remark that the potential kernel associated to $h = 1_{[0,1]}$ is given by

$(\forall f \in L_f^1(\mathrm{d}x))$

$$Wf = \begin{cases} c + (1-2x)\displaystyle\int_{-\infty}^{x} f(y)\,\mathrm{d}y + \int_{x}^{0} (1-2y)f(y)\,\mathrm{d}y, & x < 0, \\ c' + \displaystyle\int_{1}^{x} f(y)(2y-1)\,\mathrm{d}y + (2x-1)\int_{x}^{+\infty} f(y)\,\mathrm{d}y, & x > 1. \end{cases}$$

(We do not mind of $x \in [0, 1]$.)

We obtain with this result that a function f of $L^1_+(\mathrm{d}x)$ is

$(+\infty)$-strongly special if and only if $x \to xf(x)$ is integrable on $[0, +\infty[$,

$(-\infty)$-strongly special if and only if $s \to xf(x)$ is integrable on $]-\infty, 0]$.

For example the function f given by

$$f(x) = \begin{cases} 0 & \text{on }]-\infty, 1], \\ \dfrac{1}{x^2} & \text{on } [1, +\infty[\end{cases}$$

is $(-\infty)$-strongly special and not special.

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